Handle Attachments

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Working from [Kup], [Kos93]. Unfortunately to embed these attachments we will often need to go one dimension higher, and therefore it will be impossible to draw many handle attachments.

1 Handle Attachments

Let M^m be a smooth manifold with boundary and $\varphi : \partial D^i \times D^{m-i} \hookrightarrow \partial M$ a smooth embedding. Then

$$M + (\varphi) \ := \ M \cup_\varphi (D^i \times D^{m-i})$$

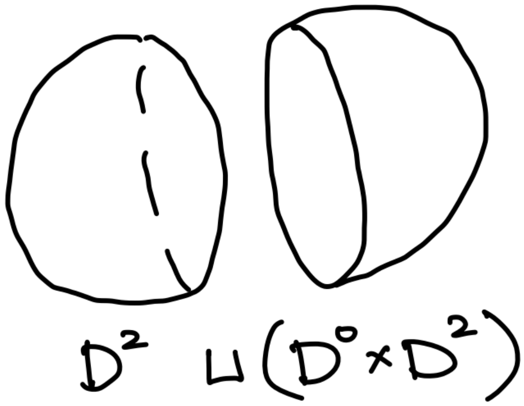
is a smooth manifold, said to be the result of attaching a handle to M. The subset $D^i \times D^{m-i}$ is called the i-*handle* or handle of index i. $D^i \times \{0\}$ is called the *core*, the boundary of the core is called the *attaching sphere*. $\{0\} \times D^{m-i}$ is called the *co-core* and its boundary is called the *transverse sphere*.

1.1 Attaching handles to D^2

First we consider the different embeddings into $\partial D^2 = S^1$.

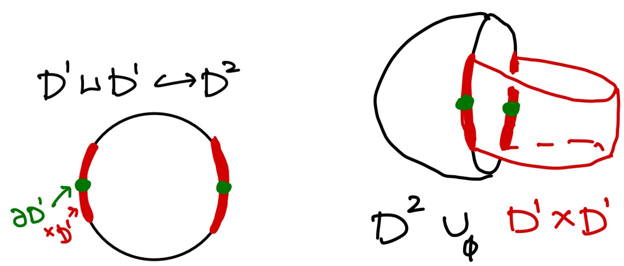
1. (Zero Handle). $\partial D^0 \times D^2 = \emptyset \times D^2 = \emptyset$. There is a unique map given by the empty set itself. It picks out no points and therefore glues nothing.
2. (One Handle). $\partial D^1 \times D^1 = \{\pm 1\} \times D^1 = D^1 \sqcup D^1$
3. (Two Handle). $\partial D^2 \times D^0 = S^1 \times * = S^1$

Attaching a zero handle therefore results in simply the disjoint union

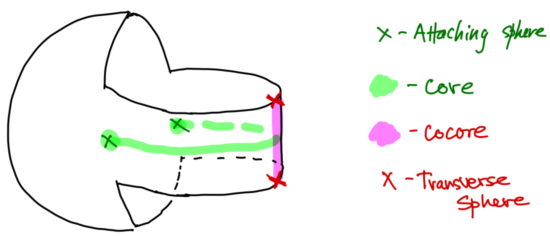


It is clear that this is always the case and so we will ignore this case going forward.

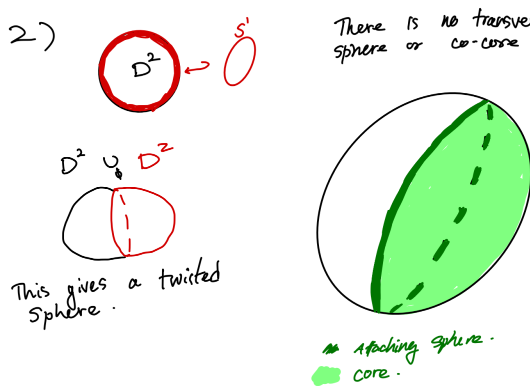
For attaching a one handle we have the following depicting an embedding Note that because it is



an embedding the two sides must be disjoint like this. We can see the cores



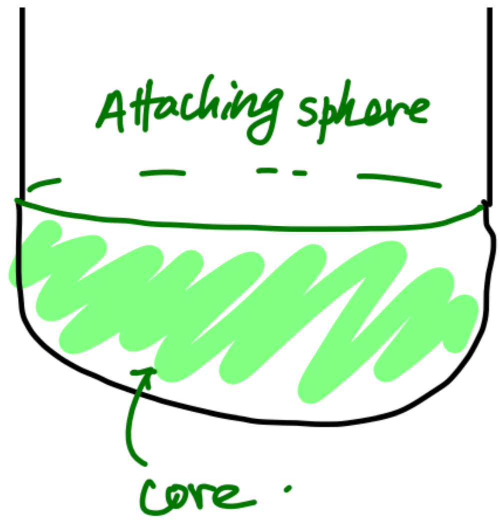
Finally attaching a two handle gives us our twisted sphere constructions We could have *twisted*



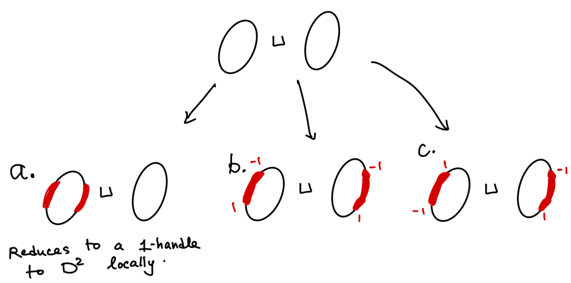
the handle attached here (and we show what happens in the next example), however if we orient the boundary of both the attached handle and the base manifold then such twists would not agree. Note that there is only the choice of one twist, as if we twist an even number of times we can isotope back to the embedding above.

1.2 Attaching Handles to a Cylinder

Again we start by looking at the embeddings into $S^1 \sqcup S^1$. Attaching a two handle is the same as attaching a disk to the end of the cylinder. This is locally the same as the attaching a two handle to the disk and so we just include a picture. More interesting are the embeddings $D^1 \sqcup D^1 \to S^1 \sqcup S^1$

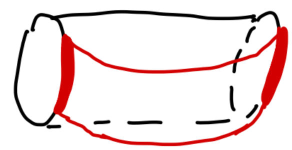


Since (a) is locally the same as the D^2 case we can skip it. The other case is like connecting a ribbon



between the two ends. We can obtain more twists by isotoping the attachments around the S^1 on the boundary:

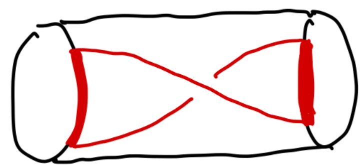
b.

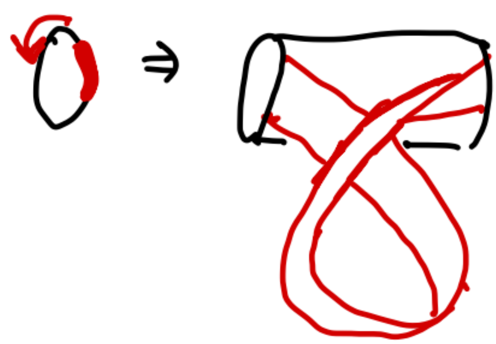


$(S^1 \times D^1) \cup_\phi (D^1 \times D^1)$

The attatching spheres & cores are clear from the D^2 case.

C.





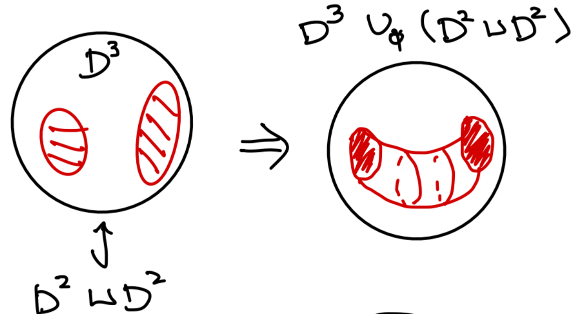
1.3 Attaching Handles to D^3

Now the boundary is an S^2 and we need to look at 3 handles as well. So the relevant embeddings are

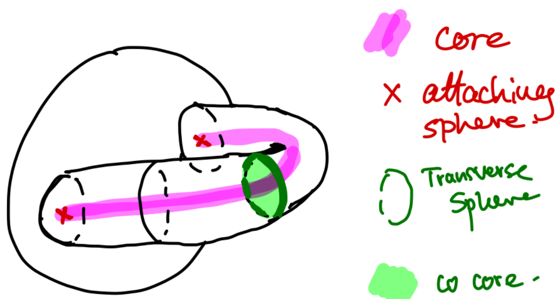
1. (One Handle). $\partial D^1 \times D^2 = \{\pm\} \times D^2 = D^2 \sqcup D^2$
2. (Two Handle). $\partial D^2 \times D^1 = S^1 \times D^1$ a cylinder
3. (Three Handle). $\partial D^3 \times D^0 = S^2$

Attaching a three handle in this case is very similar to attaching the two handle so lets skip it. Unfortunately these are really the last examples that we can really see, everything else is either higher dimensional or locally given by these handles.

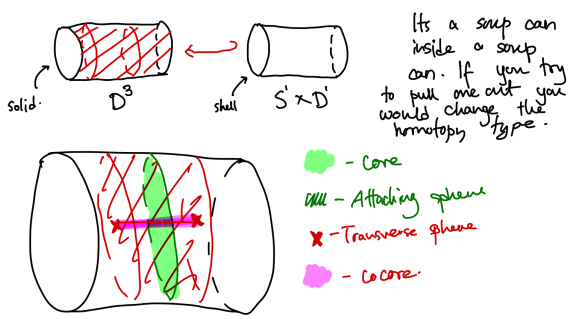
Attaching a one handle looks like this



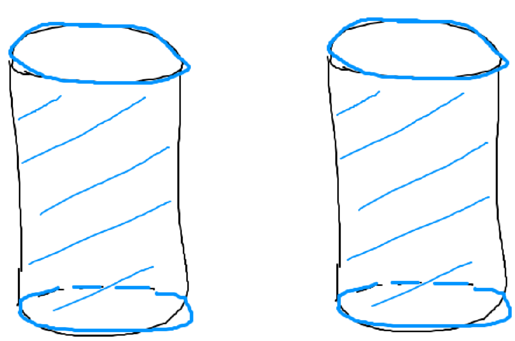
and the cores look like



If we try to attach a two handle then already we cannot draw it. The sketch is something like:



It might be better to draw the two D^3 next to each other with the blue parts being identified



Clearly each D^3 has the two end peices D^2 as the boundary, and we glue these boundary peices along *their* boundary S^1, thus the boundary of the resultant gluing is given by

$$(D^2 \cup_f D^2) \sqcup (D^2 \cup_{f'} D^2) = S^2 \sqcup S^2$$

So it is clear we have changed the manifold up to diffeomorphism. If we look at the space up to homotopy then we claim that we have added a nontrivial element of π_1. First (and this is always true) we can deformation retract the attched handle onto its core, in this case a D^2, then when we glue this in it is along its boundary S^1, we can see this as happeing say on one end of the initial cylinder. Thus we have clearly added a space between this "cap" and the body of the can.

Finally we can deformation retract the first can onto its coundary and be left with the homotopy type of S^2.

2 Handle Decompositions

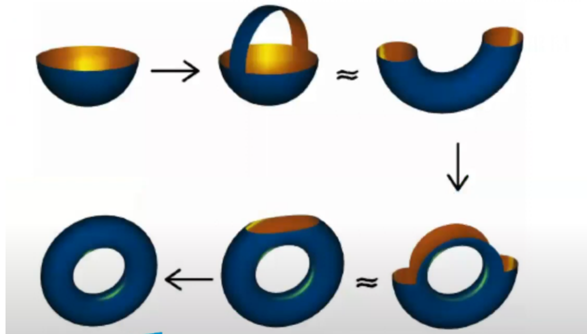
The starting point is the following theorem, proved from the existence of Morse functions

Theorem ([Kup], Cor 14.2.3)**.** *Every compact smooth manifold admits a handle decomposition, that is as a sequence of attaching handles to $\emptyset$.*

Note that the reference proves it for closed manifolds however there is an extension to manifolds with boundary.

Example (S^n)**.** *We know that every sphere is given by the twisted sphere construction. If we attach a single n handle to $\emptyset$ then this is just the disjoint union $\emptyset \sqcup D^n = D^n$. We have seen then how attaching another n handle will give us a twisted sphere (at least in the $n = 2$ case).*

Example (Torus)**.** *Start with a 2 disc (attached as a 2-handle to the empty set if you like). Then we can attach a one handle in the way depicted above. Notice that this is diffeomorphic to a cylinder by pushing in the handle and stretching out the open discs. Taking the cylinder we can then perform another one handle attachment as in the example (a). The result is a space that is diffeomorphic to a torus with a disc cut out, one can see this by pulling up the sides of the ribbon and rounding them over. Finally we attach a two handle, a disk along the boundary, which is a sphere and we get a torus.*



This gives the general idea for all other compact surfaces as well. You attach a one handle for each of the generators of the fundamental group and then you close it off at the end with a disc.

We can see a similarity here with CW constructions and indeed the idea for handles is to give a sort of CW structure that is more convenient in the smooth category. We will take this further when we see later that the cellular cochain can be replaced by the handle cochain and then show that the resulting cohomology dealing with handle operations is the same.

Remark. The empty set is an open set inside *every* $\mathbb{R}^n$, that is it is the only manifold of *every* dimension. Thus we can get away with saying that we dont change the dimension when attaching handles or in the handle decomposition. Notice however that attaching an n handle to the empty set always results in just the n disc and so we could have begun the handle attachments of an n manifold with the n disc. Using the empty set makes the definition slightly more uniform and makes the parallel with CW decompositions more apparent.

3 Handle Rearangments

Given a decomposition then of a space

$$M + \sum (\varphi_i)$$

we have the following facts

1. If φ is isotopic to φ' then $M + (\varphi) \cong M + (\varphi')$
2. If $\text{index}(\varphi) \geq \text{index}(\phi)$ then there exists a map ψ isotopic to ϕ such that

   $$M + (\varphi) + (\phi) \cong M + (\psi) + (\varphi)$$

   note here that the addition is not commutative here and that it is important that the element with the smaller index is further to the right. This in particular implies that we can order the handle attachments according to the index.
3. If $\text{index}(\varphi) = \text{index}(\phi) + 1$ and the attaching sphere of φ intersects transversally at a single point with the transverse sphere of ϕ then

   $$M + (\phi) + (\varphi) \cong M$$

   This can also be run in reverese to add two handles to a manifold without changing the diffeomorphism type.

For (1) there is sort of no non-trivial isotopies around in our examples, however this just formalising the sort of intuitive fact that say if I drag the attaching spheres around for our one-handle attachment to the three-sphere then we get the same thing.

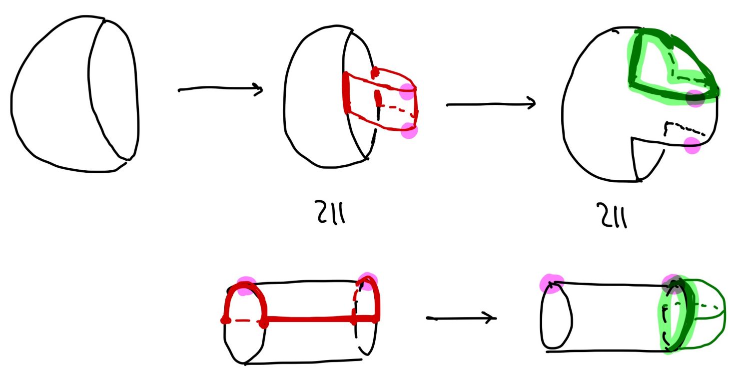
The only non-trivial example of (2) is taking the cycliner. First we attach a two-handle by capping off one end. Then we attach say a one handle to the other end. Then this states that we could have interchanged these (clear).

Lets try to cancel some handles using the previous examples.

Example (One that works)**.** *We claim that*

$$D^2 + (\textit{one handle}) + (\textit{two handle}) \cong D^2$$

The attachment of the one handle as we argued before gives a cylinder, then attaching a two handle caps off the cylinder and so we are back at the disc.

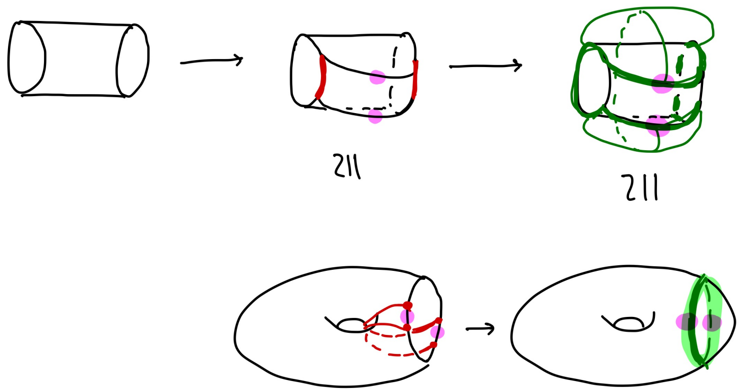


We can see in the final product the intersection of the attaching sphere of the two handle (green highlight) with the transverse sphere of the one handle (pink dots). The intersect at only a single point, and that is the entire manifold for one of them so it is clearly transverse.

Example (One that doesnt)**.** *We claim that*

$$D^1 \times S^1 + (\textit{one handle}) + (\textit{two handle}) \cong \textit{torus} \ncong D^1 \times S^1$$

This was exacly our presentation of the torus above. It is clear that the torus is not the cylinder (different fundamental group for example).



We can see that intersection of the attaching sphere of the two handle (green highlight) with the transverse sphere of the one handle (pink dots) is in two places.

Remark. A special case of handle cancelation of discs (two disc example above) was used by Hatcher to prove the Smale conjecture.

References

[Kos93] Antoni A. Kosinski. *Differential manifolds.* Number v. 138 in Pure and applied mathematics. Academic Press, Boston, 1993.

[Kup] Alexander Kupers. Lectures on diffeomorphism groups of manifolds, version February 22, 2019.